

## OPTIMAL WEIGHTLIFTING

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In international weightlifting competition there are two categories of lifts: the "snatch" and the "clean-and-jerk". A lifter's final score is the sum of his scores in these two categories. In each category the lifter has three attempts, and his "score" is the weight of his best lift. If he fails in an attempt, he cannot select a lower weight on a subsequent lift in the same category. Two problems are explored here: selection of lift weights and methods for updating that selection as more information becomes available during the competition. Several numerical examples are included.

### 1. Introduction

There are two categories of lifts in international weight lifting – first the "snatch" and then the "clean-and-jerk" (in that order). In the snatch, the lifter must take the weight in a single motion from the ground to above his head, arms extended. In the clean-and-jerk, the lifter first "cleans" the weight (brings it up to his chest) then thrusts it above his head, or "jerks" it. Each lift category is run as a separate competition; however, one's overall score is the sum of the separate scores in the two categories.

In each category the lifter faces the problem of selecting weights. A competitor has a maximum of three lift attempts in each category. His score in that category is the weight he lifts on his best lift. He cannot, however, go down in his selected weight. Once a weight is selected for a lift, the weight chosen on the next lift must be at least as high. A lifter who attempts too little may have almost no chance of winning; a lifter who attempts too much may have a significant chance of "bombing out" (scoring zero).

This paper considers both the problem of optimizing performance within a single lift category and across the two lift-categories. In addition, the important problem of updating strategy during competition is considered: if a competitor bombed out of the snatch in a particular competition, one's strategy in the clean-and-jerk should incorporate that information.

The weight selection problem demands explicit consideration of the value of scoring a certain weight as well as the chances of making the lift. These considerations are developed in the next section.

Game-theoretic and team-strategy issues are not considered here; the strategy of an individual in a competition group that is indifferent to his performance is used for our analysis. This is a simplification of the problem in which other competitors influence lift probabilities and coaches exert "team-strategy" pressure. The following analysis should provide at least norms or guidelines for decisions which can be modified by a coach or individual lifter to suit the specifics of a meet situation.

**2. The general model**

We refer here to the partial decision tree in Figure 1. The areas occupied by X's and Y's indicate decision points; the splits or forks are chance nodes.

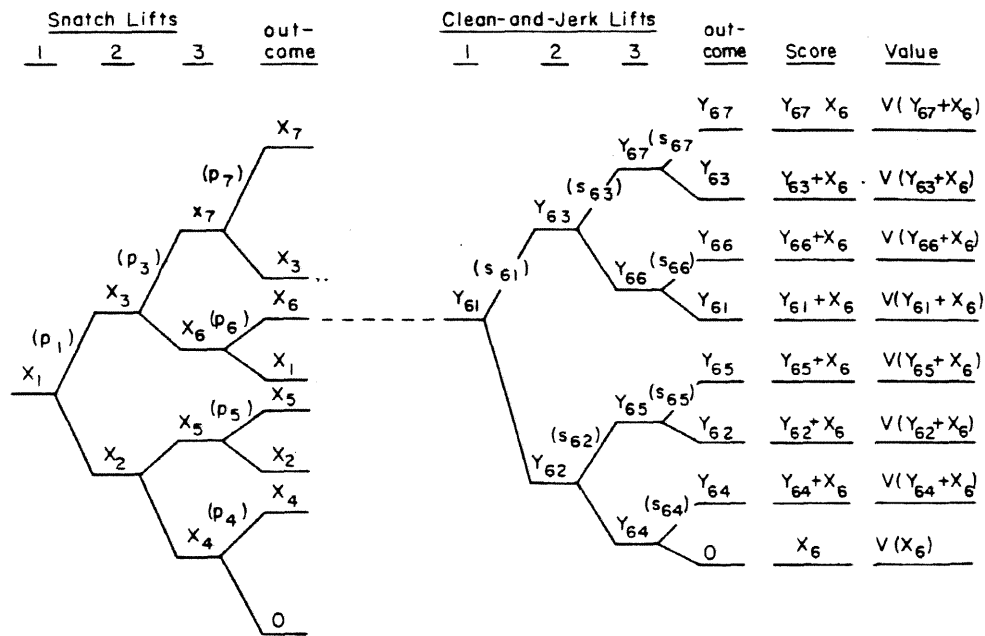


Figure 1. Weightlifting partial decision tree.

A lifter moves to the high branch after a chance node if he is successful in the lift; if he misses, he passes to the lower branch.

Define the following:

- $X_i$  = weight selected for snatch-lift  $i$  in Figure 1,  $i = 0, \dots, 7$ .
- $Q_i$  = probability that score in snatch competition is  $X_i$ ,  $i = 0, \dots, 7$ .
- $p_i$  = probability of lifting  $X_i$  in Figure 1.

$Y_{ij}$  = weight selected for clean-and-jerk lift  $j$ , given snatch-weight =  $X_i$ ,  
 $j = 0, \dots, 7$ .  $Y_{i0} = 0$ ,  $i = 0, \dots, 7$ .

$s_{ij}$  = unconditional probability of lifting  $Y_{ij}$ .

$R_{ij}$  = probability that the final score is  $Y_{ij} + X_i$ .

Assuming independence,  $Q_i$  is displayed in terms of  $\{p_i\}$  in Table 1 and  $R_{ij}$  is displayed in terms of  $\{Q_i\}$  and  $\{s_{ij}\}$  in Table 2.

| Table 1<br>Snatch-lift outcome probabilities |                         | Table 2<br>Total lift outcome probabilities |                                     |
|--|-------------------------|---|-------------------------------------|
| $i$  | $Q_i$                   | $J$   | $R_{ij}$ , $i = 0, \dots, 7$        |
| 0  | $(1-p_1)(1-p_2)(1-p_4)$ | 0   | $(1-s_{i1})(1-s_{i2})(1-s_{i4})Q_i$ |
| 1  | $p_1(1-p_3)(1-p_6)$     | 1   | $s_{i1}(1-s_{i3})(1-s_{i6})Q_i$     |
| 2  | $(1-p_1)(p_2)(1-p_5)$   | 2   | $(1-s_{i1})(s_{i2})(1-s_{i5})Q_i$   |
| 3  | $p_1 p_3 (1-p_7)$       | 3   | $s_{i1} s_{i3} (1-s_{i7})Q_i$       |
| 4  | $(1-p_1)(1-p_2)p_4$     | 4   | $(1-s_{i1})(1-s_{i2})s_{i4}Q_i$     |
| 5  | $(1-p_1)p_2 p_5$        | 5   | $(1-s_{i1})s_{i2} s_{i5} Q_i$       |
| 6  | $p_1(1-p_3)(p_6)$       | 6   | $s_{i1}(1-s_{i3})s_{i6}$            |
| 7  | $p_1 p_3 p_7$           | 7   | $s_{i1} s_{i3} s_{i7}$              |

If we let  $V(X_i + Y_{ij})$  = value or utility associated with a particular lift-sum, then we can formulate the general problem as follows:

$$(1) \quad \begin{aligned} & \text{find } \{X_i\}_{i=1}^7, \{Y_{ij}\}_{i=1}^7, j=1 \text{ to } 7 \\ & \max Z = \sum_{j=0}^7 \sum_{i=0}^7 V(X_i + Y_{ij}) R_{ij} \end{aligned}$$

subject to competition rules:

$$(1a) \quad \begin{cases} X_1 \leq X_2 \leq X_4 \\ X_3 \leq X_6 \end{cases}$$

$$(1b) \quad \begin{cases} Y_{i1} \leq Y_{i2} \leq Y_{i4} & i = 0, \dots, 7 \\ Y_{i3} \leq Y_{i6} & i = 0, \dots, 7 \\ X_0 = 0 \\ Y_{i0} = 0 & i = 0, \dots, 7 \\ X_i \geq 0, \quad Y_{ij} \geq 0 & i = 1, \dots, 7, \quad j = 1, \dots, 7. \end{cases}$$

In addition, scoring rules imply:

$$(1c) \quad \begin{cases} X_1 < X_3 < X_7 \\ X_2 < X_5 \\ Y_{ij} < Y_{i3} < Y_{i7} & i = 0, \dots, 7 \\ Y_{i2} < Y_{i5} & i = 0, \dots, 7. \end{cases}$$

The above is a general nonlinear program. When information about the form of the objective function is available, the relatively small number of variables (56) and the simple nature of the constraints makes it small enough to be solved by existing codes. The structure of the tree in Figure 1 suggests a dynamic programming solution strategy. However, the nature of the constraint structure makes such a strategy awkward in general.

### 3. Evaluation of lift probabilities

One could evaluate a lifter's previous performances to attempt to assign probabilities to various lift-weights. However, this assumes that we can take account of day-to-day variations in performance and relate that to his performance on a particular day. Clearly, any estimate of lift probabilities would have to be modified to reflect the lifter's mental and physical state on a particular day. The procedure we suggest, basically asking the lifter what he thinks he can lift, assumes he has enough knowledge of his past performance to incorporate that knowledge in an estimate of his performance on a particular occasion. Thus we shall "ask the horse", although, experience with particular lifters may call for modifications.

One simple method for obtaining this information from a lifter would be to ask for three points on a (bounded) probability-of-lift curve as follows:

Question 1: "What is the maximum weight you feel certain you can lift?"  
Call this  $L$ .

Question 2: "What is the most weight you feel you have the slightest chance of lifting?" Call this  $U$ .

Question 3: "What weight do you feel you have a 50-50 chance of lifting?"  
Call this  $M$ .

If we then assume that the lifter's probability-of-lift curve is well behaved, we might approximate it by a quadratic function between  $L$  and  $U$  as follows:

Assume  $p_i(X_i)$  = probability of lifting  $X_i$ , is of the form

$$(2) \quad p_i(X_i) = \begin{cases} 1 & X_i \leq L; \text{ or } U > X_i > L \\ & \text{and } f(z) > 1 \\ f(z) = az^2 + bz + c & U > X_i > L \text{ and } 1 \geq f(z) > 0 \\ 0 & X_i \geq U; \text{ or } U > X_i > L \\ & \text{and } f(z) < 0 \end{cases}$$

for all  $i$

where  $z = (X - L)/(U - L)$  = proportion of a lifter's "weight-range" attempted. If we let  $k_0 = (M - L)/(U - L)$ , then it can be shown that  $a =$

$(1/2 - k_0)/(k_0 - k_0^2)$ ,  $b = (k_0^2 - 1/2)/(k_0 - k_0^2)$  yielding  $f(z) = 1$  for  $X_i = 0$ ,  $f(z) = 0$  for  $X_i = L$  and  $c = 1$  in (2). We have assumed for simplicity that this function is the same for any lift; clearly if a lifter were able to quantify a different function for each  $i$  in (2), a set of probability functions could be estimated and used in (1).

#### 4. Objectives

Many methods have been suggested for assessing utility measures (see Raiffa [3], e.g.). Several with some intuitive appeal are as follows:

*Expected Weight Maximization:* This assumes a lifter's utility is linearly related to the amount he lifts. A problem exists: there is (relatively) a very high negative utility associated with bombing out. This negative utility may be higher than that which exists in lifters' minds.

*Weight-Range Maximization:* Here it is assumed the lifter wishes to maximize the proportion of his "weight range" he achieves; i.e., a lifter who lifts 350 pounds with  $U = 400$ ,  $L = 300$ , scores 0.5. This suggests more risk-prone strategies, since the utility of 300 to this lifter is 0, which is the same as the utility of his bombing out. Hence he would be indifferent between 300 and 0, while an expected weight maximizer clearly would not. This may or may not be the case but observation of liftings suggest some (at least psychological) penalties for bombing out. One might attempt to assess that penalty directly and include it as  $V(0)$ . In that case, "Weight-Range" and Weight Maximization could be made to imply the same policy. (They would differ only by a linear transformation, which would have no effect on optimal policies.)

*Medal Winning:* This, after all, is what competition is about. One method of assessment of chances of winning medals calls for analysis of prior scores by competing lifters. Assume  $N$  lifters are involved in the competition, and an evaluation of lifter  $k$ 's past performance yields a snatch-density, potentially of the mixed variety,  $dF_x^k(x)$  and a clean-and-jerk density  $dF_x^k(y)$ . Thus, the following (prior) information is available:

| Lifter | Snatch density | Clean-and-jerk density |
|--------|----------------|------------------------|
| 1      | $dF_x^1(x)$    | $dF_x^1(y)$            |
| 2      | $dF_x^2(x)$    | $dF_x^2(y)$            |
| .      | .              | .                      |
| .      | .              | .                      |
| .      | .              | .                      |
| $N$    | $dF_x^N(x)$    | $dF_x^N(y)$            |

Let:

$Z_i$  = random variable, score lifted by  $i$ th competitor  
 =  $X^i + Y^i$  where  $X^i$  =  $i$ th competitor's snatch score,  
 $Y^i$  =  $i$ th competitor's clean-and-jerk score.

$dF_{Z^i}(z)$  = (mixed) probability density of score of  $i$ th competitor  
 = convolution of  $X^i$  and  $Y^i$   
 =  $\int_0^z dF_{X^i}(y) dF_{Y^i}(z - y)$

$W_i$  =  $i$ th highest weight lifted;  $W_1$  = lowest weight,  $W_N$  = winning weight.

Assuming that the weights are independently distributed, the joint probability density of the  $N$  order statistics and marginal density of the  $r$ th highest weight lifted can be calculated in a straightforward manner.

For example, if an individual is only satisfied with first place,

$$dG_{W_N}(w_N) = \sum_{i=1}^N \prod_{\substack{j=1 \\ j \neq i}}^N F_{X^j}(w_N) dF_{X^i}(w_N)$$

and

$$V(T) = \int_0^T dG_{W_N}(w_N) = \text{"value" of lift-total } T \text{ in terms of probability of winning.}$$

We have assumed we can analyze snatch lifts and clean-and-jerk lifts independently here. This assumption allows simpler analysis of past lifts to form densities than would dependent, joint distributions (which could well be beyond our ability to analyze in any case).

### 5. Updating during competition

Suppose that (competitive) lifter  $i$  has already lifted  $\bar{X}^i$ . It no longer makes sense to consider events including  $X^i < \bar{X}^i$ ; thus lifter  $j$ 's lift-density should be updated. The updating procedure is simple; using the rule of conditional probability.

$$dF_{X^j|X^i > \bar{X}^i}^j(x) = \begin{cases} 0 & x < \bar{X}^i \\ \frac{dF_{X^j}(x)}{\int_{\bar{X}^i}^{\infty} dF_{X^j}(x)} & x > \bar{X}^i. \end{cases}$$

Then the updated  $dF_{X^j}$  (or  $dF_{Y^j}$ ) replaces the prior distribution. Then the objective function and the lift-strategy can be updated.

Suppose also that, after lifting  $X_1$ , a lifter wishes to update his probability-of-lift curve, say, by increasing his 50-50 weight by 10 pounds. Then a new,

optimal strategy, incorporating this new probability-of-lift information in the calculation of the  $\{p_i\}$  should be developed.

The above procedures allow updating and rerunning of the model during a competition, incorporating information available at that time so that best decisions can be made.

## 6. An independence theorem for lift optimality

Under what conditions can a lifter in the second category, the clean-and-jerk, neglect the results of his first category lift in choosing an optimal strategy? That question is treated in the following:

**Theorem.** *If  $V(X_i + Y_{ij})$  is a separable function of  $Y_{ij}$  and  $X_i$ , then clean-and-jerk strategy is independent of snatch results.*

**Proof.** See Lilien [2].

A particular case of this theorem\* is the following:

**Lemma.** *When the objective is expected weight maximization or weight-range maximization, 1st and 2nd category lift strategies are independent.*

**Proof.** As shown in [2], these objectives lead to separable  $V(X_i + Y_{ij})$ .

Thus, we have the useful result that if our objective is one of the above types, we need only concern ourselves with the current competition and not be concerned about other category performance.

## 7. Some numerical examples

The structure of Figure 1 suggests dynamic programming as a solution strategy for this class of problems. However, explicit consideration of constraints (1a) makes that solution procedure rather messy in general. Simple objectives and a one lift, two attempt simplification of the problem allow analytically tractable results which are of some intuitive importance.

We now consider the example in Figure 2 using the objective: weight range maximization. Here the probability of lifting weight  $X$  is  $(U - X)/(U - L)$  and the utility of that lift is  $(X - L)/(U - L)$ . (We have assumed that a lifter's 50-50 weight =  $(U + L)/2$  here.) We then wish to maximize the expected proportion of a lifter's weight-range which is realized. Backward induction leads to the solution  $X_1 = X_2 = L + 0.525(U - L)$ ,  $X_3 = L + 0.763(U - L)$ .

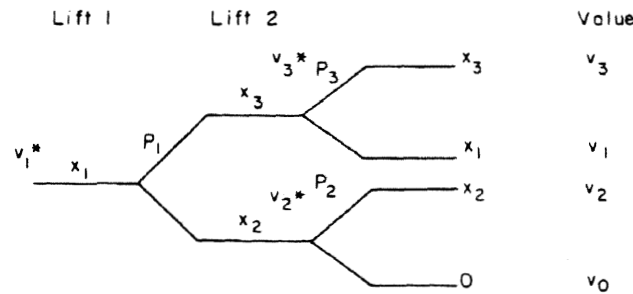


Figure 2. Two-lift simplified tree.

If we now consider expected weight maximization with linear probability structure, we find that probability of lifting any weight is the same as above, but the utility structure differs. Here  $V(X) = X$  for any  $X$ . In this case it can be shown that

$$X_1 = X_2 = \begin{cases} L & \text{if } U/L < 5 \\ \frac{6U + \sqrt{36U^2 + 60(U^2 - 8UL)}}{30} & \text{if } U/L \geq 5. \end{cases}$$

and

$$X_3 = \frac{U + X_1}{2}$$

We might ask at this point: who takes greater risks – the expected weight or proportion maximizer? Table 3 gives the lifting strategies for this two lift case for the weight maximizer for various values of  $U/L$ .

Comparing the optimal lifts here with those in Section 5.1, we see that as  $U/L \rightarrow \infty$ , optimal weight selections for the weight maximizer monotonically increase and converge to those of the proportion maximizer.

Table 3  
Optimal lifts – weight maximization

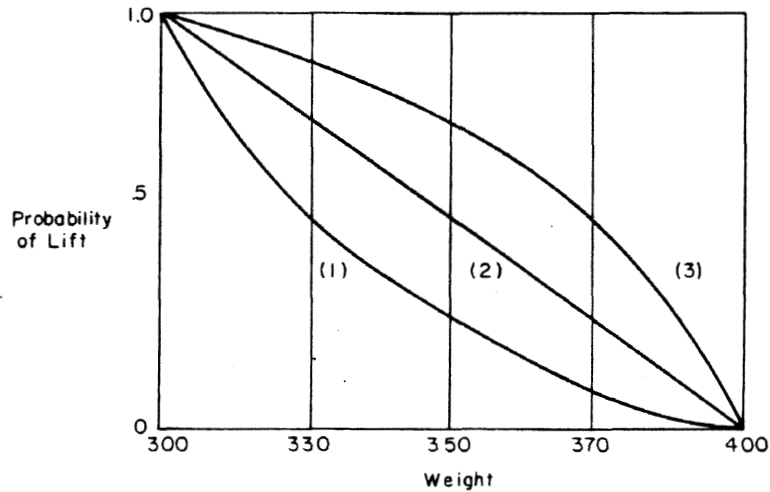
| $U/L$    | $X_1 (= X_2)$ | $X_3$             |
|----------|---------------|-------------------|
| $< 5$    | $L$           | $\frac{U + L}{2}$ |
| $5$      | $0.2U (= L)$  | $0.6U$            |
| $10$     | $0.434U$      | $0.717U$          |
| $20$     | $0.487U$      | $0.744U$          |
| $\infty$ | $0.525U$      | $0.763U$          |

Thus the proportion maximizer is more risk prone than the weight maximizer in this case.



It is most convenient to study the general case numerically, using a nonlinear optimization method. In this section we will give a variety of examples to illustrate the sensitivity of optimal policies to changes in objectives and input parameters.

For simplicity consider an individual whose value of  $U = 400$ ,  $L = 300$  and  $M = 330, 350$  or  $370$  (see Section 3), either in the snatch or the clean-and-jerk competition. Probability curves for these parameter values are included in Figure 3.



- (1) = 50-50 weight = 330  
 (2) = 50-50 weight = 350  
 (3) = 50-50 weight = 370

Figure 3. Probability structure for case studies.

Three objectives were considered:

A = Maximizing expected proportion of lift-range.

B = Maximizing expected weight.

C = Maximizing chances of medal winning given a score of  $X^*$  in the snatch.

For objective C, it was assumed that on the basis of prior data analysis, one's chances of winning a medal =

$$T = \begin{cases} 0 & \text{for clean-and-jerk Weight Total (WT)} \leq 350 \\ \frac{WT - 350}{100}, & 350 < WT < 450 \\ 1 & WT \geq 450 \end{cases}$$

i.e., the minimum weight needed to win a medal is a random variable uniformly distributed between 350 and 450. There is no inherent difficulty in making this function more complicated since the solution was calculated numerically anyway; this simple example was used for ease of exposition.

The numerical solutions listed in Table 4 were calculated using RAC's nonlinear programming package SUMT [1]. Each case took about six seconds of CPU time to run on an IBM model 370/165. The program is currently in a batch mode although it could be adapted for real time use.

Table 4  
Optimal lift sequences

| Case # | Optimal weights |       |       |       |       |       |       | Objective* | Probability curve** | Expected value of objective |
|--------|-----------------|-------|-------|-------|-------|-------|-------|------------|---------------------|-----------------------------|
|        | $X_1$           | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ |            |                     |                             |
| 1      | 338             | 338   | 360   | 338   | 360   | 360   | 374   | A          | 1                   | 33%                         |
| 2      | 355             | 355   | 378   | 355   | 378   | 378   | 389   | A          | 2                   | 51%                         |
| 3      | 364             | 364   | 383   | 364   | 383   | 383   | 392   | A          | 3                   | 67%                         |
| 4      | 300             | -     | 336   | -     | -     | 336   | 358   | B          | 1                   | 326                         |
| 5      | 315             | 315   | 359   | 315   | 357   | 359   | 380   | B          | 2                   | 342                         |
| 6      | 339             | 339   | 373   | 339   | 373   | 373   | 387   | B          | 3                   | 361                         |
| 7      | 367             | 367   | 379   | 367   | 379   | 379   | 386   | C          | 1                   | 6%                          |
| 8      | 375             | 375   | 388   | 375   | 388   | 388   | 394   | C          | 2                   | 16%                         |
| 9      | 378             | 378   | 390   | 378   | 390   | 390   | 395   | C          | 3                   | 24%                         |

\*A = Max expected proportion of weight range

B = Max expected weight

C = Max probability of medal where

$$\text{Prob} = \begin{cases} 0, & (\text{WT}) \leq 350 \\ \frac{\text{WT} - 350}{100}, & 350 < \text{WT} \leq 450 \\ 1, & \text{WT} > 450 \end{cases}$$

\*\*See Figure 3 for associated curve.

A single lift "solution" from Table 4 is read as follows: Line 1 indicates that the lifter is an expected proportion maximizer whose 50-50 weight ( $M$ ) = 330. His optimal first lift weight is 338 pounds ( $X_1$ ). If he makes that lift his next lift should be 360 ( $X_3$ ) pounds; if he misses, 338 ( $X_2$ ) again and so on in accordance with the notation in Figure 1. On the average he will lift 33% of his weight-range.

Line 4 has blanks under  $X_2$ ,  $X_4$ ,  $X_5$ ; this is due to the fact that  $X_1 = 300$ , a certain event, making  $X_2$ ,  $X_4$ ,  $X_5$  events of probability zero.

Now let us consider a two-lift example for a lifter with probability curve (1) in the snatch, (3) in the clean-and-jerk and who wishes to maximize expected weight (Objective B). According to the theorem, the snatch and clean-and-jerk decisions can be made independently as indicated in Figure 4; i.e. the lifter starts his snatch lifts at 300, increases to 336 then to 358 (if successful). He then starts the clean-and-jerk competition at 339, etc.

Lift probabilities have been assumed constant across lifts; they could of course have been updated as in Section 5. It is also simple (though entailing more computer time) to calculate a full set of the 56 lift values (in eq. (1)) under conditions of non-independence.

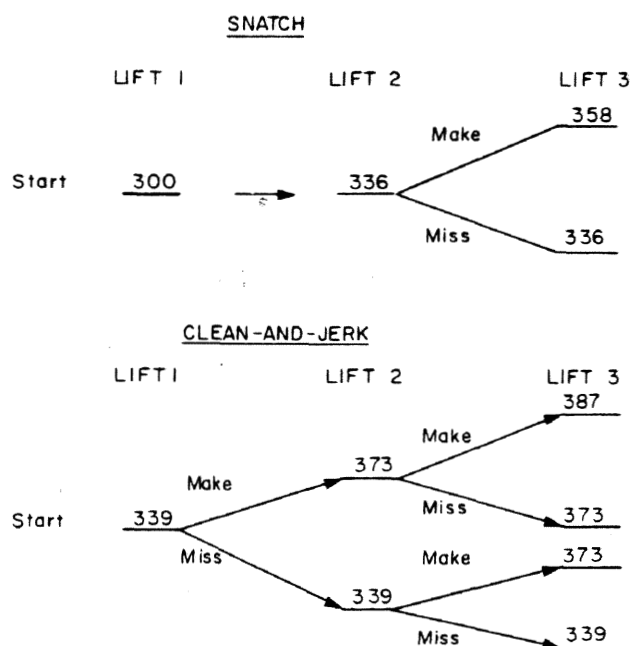


Figure 4. Two-lift example.

## 8. Conclusions and uses

An analytic procedure has been proposed here to aid in the selection of optimal weights for a weightlifter. The weight selection policies have been shown to be quite sensitive to a lifter's objectives and utilities, indicating that careful consideration of those utilities is essential in any application. Cases 1, 4 and 7 in Table 3, for example, are identical except for objectives and they result in vastly different policies.

The numerical procedure has been found to be efficient enough to allow for real-time operation at an actual meet. Thus a lifter (whose utilities have been determined beforehand) could wait until just before he was ready to

select his lift weight to answer the lift probability questions in Section 3. On the basis of his first or second lift he could then come back to the computer terminal and update his probabilities in time for his next lift. He could also enter competitive, current scores to update his objective function. He would then receive an updated, "best" lift suggestion for the next round of competition.

A more likely mode of implementation might be to generate a book of strategy-tables (as in Table 4) at reasonable intervals which could be used for real-time interpolation at a meet.

### References

- [1] Fiacco, A.V. and G.P. McCormack, 1968, *Nonlinear Programming: Sequential Unconstrained Minimization Techniques* (Wiley & Sons, New York).
- [2] Lilien, G.L., December 1974, *Optimal Weightlifting*, Sloan School of Management Working Paper 757-74.
- [3] Raiffa, H., 1970, *Decision Analysis*, (Addison-Wesley, Reading, Mass.).